Linear response to perturbation of nonexponential renewal process: A generalized master equation approach

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The work by Barbi, Bologna, and Grigolini [Phys. Rev. Lett. **95**, 220601 (2005)] discusses a response to alternating external field of a non-Markovian two-state system, where the waiting time between the two attempted changes of state follows a power law. It introduced a new instrument for description of such situations based on a stochastic master equation with reset. In the present Brief Report we provide an alternative description of the situation within the framework of a generalized master equation. The results of our analytical approach are corroborated by direct numerical simulations of the system.

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This Brief Report is motivated by the recently published Ref. [1], which discusses an important problem of the response of a non-Markovian system to a time-dependent field. It also introduces a new instrument for the description of such situations based on a Markovian but stochastic master equation with reset. Let us consider a two-state model with a particle jumping between the two sites. A particle arriving at a site $i=1,2$ at time t' stays there for a time t distributed with the probability density function (PDF) $\psi(t)$ before the next attempt to jump is made. The probability w_{ij} that the particle really jumps from i to j is modulated by the force $f(t)$:

$$w_{12}(t) = \frac{1}{2}[1+\varepsilon f(t)],$$

$$w_{21}(t) = \frac{1}{2}[1-\varepsilon f(t)]. \quad (1)$$

This model corresponds to the one of Ref. [1] and to what is called a "phenomenological approach" in Ref. [2]. Two different types of waiting-time distributions (WTDs) have to be distinguished: the ones possessing a first moment and the ones whose first moment is absent. The WTD PDFs discussed in Ref. [1] were of the form of power laws $\psi(t) \propto t^{-\mu}$ with $2<\mu<3$ for the first type and $1<\mu<2$ for the second type. Two-state systems with WTDs of the first type show at long times a behavior similar to those of Markovian two-state systems (as also discussed in Ref. [3]); systems with WTDs of the second type are special: the linear response to the external field is nonstationary and dies out in the course of time.

The situation discussed in Ref. [1] is very close to models of continuous-time random walks (CTRWs), but differs from the typical CTRW problem in two respects: First, the transitions take place under the influence of a time-dependent force, and second, the particle does not necessarily make a jump on each attempt, but may stay where it was. Continuous-time random walks can be very effectively described using approaches based on generalized master equations. Therefore it is reasonable to give a derivation of the generalized master equation (GME) for this particular situation and to compare the results with the ones obtained in Refs. [1,2] using alternative approaches.

The derivation of the GME follows the lines of Ref. [4] (which, in its turn, generalizes the approach of Ref. [5]), where, however, the differences with respect to a simple CTRW have to be taken into account. Let us first consider a general system whose states are numbered by $k=1,2\ldots,n$ and where a change of state takes place at each attempt. The transition probabilities $W_{ij}(t)$ for a system making a jump from state i to state j are time dependent. These probabilities are normalized, $\Sigma_{j\neq i}W_{ij}(t)=1$. As in Ref. [4], the generalized master equation follows from two balance conditions, probability conservation in a given state and under transitions between different states.

The probability balance for the state k reads

$$\dot{P}_k = j_k^+ - j_k^-(t) \quad (2)$$

(where the overdot denotes the time derivative) with $j_k^{\pm}(t)$ denoting the gain and loss currents for a state. A system leaving its state k at time t either was in k from the very beginning or arrived at k at some $0<t'<t$ so that

$$\begin{aligned} j_k^-(t) &= \psi(t)P_k(0) + \int_0^t \psi(t-t')j_k^+(t')dt' \\ &= \psi(t)P_k(0) + \int_0^t \psi(t-t')[\dot{P}_k(t') + j_k^-(t')]dt', \end{aligned} \quad (3)$$

where in the second line Eq. (2) was used.

The solution to this equation is given by the integral operator

$$j_k^-(t) = \hat{\Phi}P_i(t) = \int_0^t \Phi(t-t')P_i(t')dt' \quad (4)$$

with the memory kernel given by its Laplace transform

$$\tilde{\Phi}(u) = \frac{u\tilde{\psi}(u)}{1-\tilde{\psi}(u)}. \quad (5)$$

Note that all these equations are *local*, i.e., involving only variables pertinent to the same state.

The probability conservation for transitions between different states gives the relation between the gain current in the state k and loss currents in all other states:

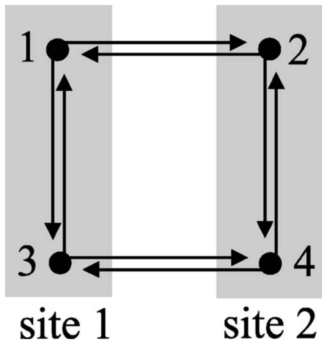

FIG. 1. The structure of transitions in a four-state model equivalent to the two-site model of Ref. [1]. The only nonzero transition probabilities are $W_{13}=W_{31}=1-w_{12}$, $W_{24}=W_{42}=1-w_{21}$, $W_{12}=W_{34}=w_{12}$, and $W_{21}=W_{43}=w_{21}$.

$$j_k^+ = \sum_{i\neq k} W_{ij}(t) j_i^- . \tag{6}$$

Inserting the corresponding expressions into the first balance equation gives a GME for $P_k(t)$:

$$\dot{P}_k(t) = \sum_{i\neq k} W_{ij}(t)\hat{\Phi}P_i(t) - \hat{\Phi}P_k(t). \tag{7}$$

Note that the integral operator $\hat{\Phi}$ does not commute with the function of time $W_{ij}(t)$; the sequence of this function and the integral operator acting only on $P(t)$ is of importance.

The two-state system at hand differs from the general scheme discussed above due to the fact that an attempted jump does not lead to a change in the system's state, but starts the waiting time anew. To adapt our general approach to this situation we assume that the attempt not leading to a jump still corresponds to a change of the state of the system, say between $k=1$ and $k=3$ for site 1 or between $k=2$ and $k=4$ for site 2; the structure of the corresponding transitions is shown in Fig. 1.

For our four-state system Eq. (7) reads

$$\dot{P}_1 = w_{21}(t)\hat{\Phi}P_2(t) + [1-w_{12}(t)]\hat{\Phi}P_3(t) - \hat{\Phi}P_1,$$

$$\dot{P}_3 = w_{21}(t)\hat{\Phi}P_4(t) + [1-w_{12}(t)]\hat{\Phi}P_1(t) - \hat{\Phi}P_3,$$

$$\dot{P}_2 = w_{12}(t)\hat{\Phi}P_1(t) + [1-w_{21}(t)]\hat{\Phi}P_4(t) - \hat{\Phi}P_2,$$

$$\dot{P}_4 = w_{12}(t)\hat{\Phi}P_3(t) + [1-w_{21}(t)]\hat{\Phi}P_2(t) - \hat{\Phi}P_4. \tag{8}$$

This auxiliary system of equations can be rewritten as a pair of equations for the probabilities $p_1=P_1+P_3$ and $p_2=P_2+P_4$ to be at sites 1 or 2, respectively, following as sums of the first and the second, and of the third and the fourth equations, respectively:

$$\frac{d}{dt}p_1(t) = -w_{12}(t)\hat{\Phi}p_1(t) + w_{21}(t)\hat{\Phi}p_2(t),$$

$$\frac{d}{dt}p_2(t) = w_{12}(t)\hat{\Phi}p_1(t) - w_{21}(t)\hat{\Phi}p_2(t), \tag{9}$$

a generalized master equation following the standard form of a master equation for a two-state Markovian system.

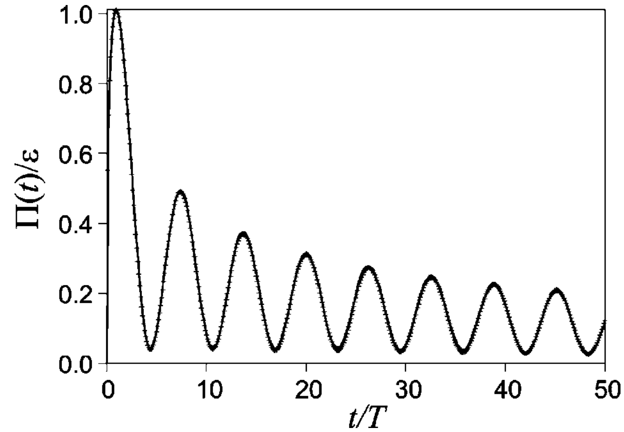

FIG. 2. Analytical result Eq. (11) (line) and results of numerical simulation averaged over 10^7 realizations (crosses). The parameters are $\mu=3/2$, $\omega=1$, and $\varepsilon=0.1$. Shown is the dimensionless quantity Π/ε as a function of dimensionless reduced time t/T.

We now follow the program of Ref. [1], and reduce these two equations to a single equation for the mean $\Pi(t)=p_1-p_2=2p_1(t)-1$, the main quantity of interest in Ref. [1]. Inserting the expressions for w_{ij}, one gets $(d/dt+\hat{\Phi})\Pi(t)=-\varepsilon f(t)\hat{\Phi}1$. For $\Pi(0)=0$ and $f(t)=\cos(\omega t)$ one gets

$$\Pi(u) = -\frac{\varepsilon[1-\psi(u)]}{u}\,\mathrm{Re}\left(\frac{\psi(u+i\omega)}{1-\psi(u+i\omega)}\right). \tag{10}$$

This equation coincides with Eq. (62) of Ref. [2]. It reproduces the asymptotic behavior found in [1] for $\psi(t)$ possessing a first moment, i.e., for $\mu>2$. However, for the aging case $1<\mu<2$ the prediction differs from Eqs. (32) and (33) of Ref. [1]: Our result Eq. (10) differs from the results of Ref. [1] by the fact that it oscillates not around zero, but around some mean which tends to zero only very slowly (here as $1/\sqrt{t}$), an effect called "Freudistic" memory in Ref. [6].

In order to check the validity of Eq. (10) we compare it with the result of direct numerical simulation of the process. We took $\psi(t)=1/\sqrt{\pi t}-e^t\,\mathrm{erfc}(\sqrt{t})$ corresponding to a long-tailed $\psi(t)$ with $\mu=3/2$ and with $T=1$. The analytical result is then given by a convolution

$$\Pi(t) = (\varepsilon/\pi)\int_0^t e^{t-t'}\,\mathrm{erfc}(\sqrt{t-t'})\cos(\omega t')/\sqrt{t'}\,dt'. \tag{11}$$

Figure 2 compares this expression with the results of numerical simulation for $\omega=1$ and $\varepsilon=0.1$.

Let us summarize our findings. We considered a situation of a two-state system with a given waiting-time distribution between the attempted changes of state, a model discussed in Ref. [1] and termed a phenomenological approach in Ref. [2]. We derived the generalized master equation describing this nonstandard situation, which reproduces part of the results of Refs. [1,2]. The validity of our generalized master equation approach to the aging situation is proved by comparison to direct numerical simulations of the corresponding system.

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